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Stability analysis of numerical methods for delay integro-differential equations

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Abstract

Stability of θ-methods for delay integro-differential equations (DIDEs) is studied on the basis of the linear equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \lambda u(t) + \mu u(t-\tau) + \kappa \int_{t-\tau}^{t} u(\sigma)d\sigma,$$

where λ, μ, κ are complex numbers and τ is a constant delay. It is shown that every A-stable θ-method possesses a similar stability property to P-stability, i.e., the method preserves the delay-independent stability of the exact solution under the condition that τ/h is an integer, where h is a step-size. It is also shown that the method does not possess the same property if τ/h is not an integer. As a result, any θ-method cannot possess a similar stability property to GP-stability with respect to DIDEs.

1. Introduction

We study stability of (2-stage) θ-methods for delay integro-differential equations (DIDEs) on the basis of the linear equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \lambda u(t) + \mu u(t-\tau) + \kappa \int_{t-\tau}^{t} u(\sigma)d\sigma, \tag{1.1}$$

where λ, μ, κ are complex numbers and τ is a constant delay. When $\kappa = 0$, the equation (1.1) coincides with the test equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \lambda u(t) + \mu u(t-\tau), \tag{1.2}$$

which was proposed by Barwell [1] to examine stability of numerical methods for delay differential equations (DDEs). As described in [1], if λ, μ satisfy

$$|\mu| < -\mathrm{Re}\,\lambda, \tag{1.3}$$

the zero solution of (1.2) is asymptotically stable for any $\tau \geq 0$. This asymptotic property is called delay-independent stability, and analogous stability properties of numerical methods are considered on the basis of the condition (1.3). For example, a numerical method for DDEs is said to be P-stable if every numerical solution to (1.2) tends to zero whenever λ, μ satisfy (1.3) and τ/h is an integer, where h is the step-size. A numerical method is said to be GP-stable if the same holds for any constant step-size.

In the last two decades, various studies were carried out concerning stability properties of numerical methods for DDEs (see, e.g., [12]). In particular, an earliest study by Watanabe and Roth [10] has revealed that every A-stable θ-method is GP-stable. To the contrary, little is known about stability properties of numerical methods for DIDEs. It is quite recent that we studied delay-independent stability of linear DIDEs [7], and even stability of θ-methods for (1.1) remains to be investigated.

By Theorem 2 of [7], the zero solution of (1.1) is asymptotically stable for any $\tau \geq 0$ if and only if λ, μ, κ satisfy

$$\lambda + \mu + \kappa\tau \neq 0 \quad \text{for any } \tau \geq 0, \tag{1.4}$$

$$z^2 - z\lambda - \kappa = 0,\ z \in \mathbb{C},\ z \neq 0 \quad \Longrightarrow \quad \operatorname{Re} z < 0, \tag{1.5}$$

$$\left| \frac{\mu z - \kappa}{z^2 - z\lambda - \kappa} \right| < 1 \quad \text{for any } \operatorname{Re} z = 0 \text{ with } z \neq 0. \tag{1.6}$$

Moreover, the conditions (1.5), (1.6) are rewritten as

$$\operatorname{Re}\lambda < 0 \quad \text{and} \quad \Big(\operatorname{Re}\lambda \operatorname{Re}(\lambda\bar{\kappa}) + (\operatorname{Im}\kappa)^2 < 0 \text{ or } \kappa = 0\Big), \tag{1.7}$$

$$\operatorname{Im}\Big[(\lambda+\mu)\bar{\kappa}\Big] = 0 \quad \text{and} \quad \Big[\ |\,\mu\,|^2 < (\operatorname{Re}\lambda)^2 + 2\operatorname{Re}\kappa \\ \text{or } \Big(\operatorname{Im}\lambda = 0,\ |\,\mu\,|^2 = (\operatorname{Re}\lambda)^2 + 2\operatorname{Re}\kappa\Big)\ \Big], \tag{1.8}$$

respectively (Sect. 3 in [7]). When λ, μ, κ are all real and $\kappa \neq 0$, these conditions are reduced to the simple condition

$$\lambda < 0, \quad \kappa < 0, \quad \mu^2 \leq \lambda^2 + 2\kappa. \tag{1.9}$$

We study stability properties of θ-methods by comparing the region determined by these conditions with a kind of stability regions of the methods.

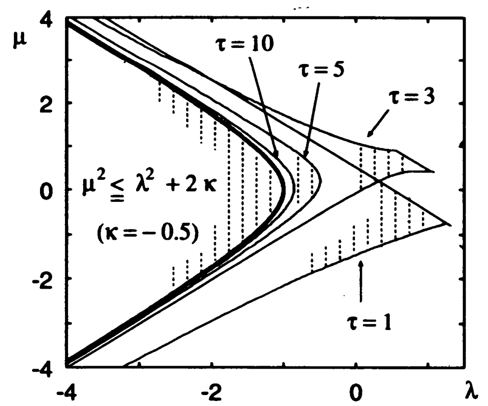

Fig. 1 Delay-independent v.s. delay-dependent stability regions

It should be noted that a considerable number of papers [2, 3, 4, 6, 9] are devoted to stability analysis of θ-methods for DDEs, which does not seem strange from a practical viewpoint. Some important instances of stiff DDEs are obtained from the space-descritization of partial functional differential equations (see, e.g., [13]). The θ-methods have practicality in such a situation.

2. Stability regions of θ-methods

Consider delay integro-differential equations (DIDEs) with a constant delay,

$$\frac{\mathrm{d}u}{\mathrm{d}t} = f\Big(t, u(t), u(t-\tau), \int_{t-\tau}^{t} g\big(t, \sigma, u(\sigma)\big)d\sigma\Big). \tag{2.1}$$

For a given step-size $h > 0$, let m be the smallest integer greater than or equal to τ/h. Then, the delay τ is represented in the form

$$\tau = (m-\delta)h, \quad 0 \le \delta < 1, \tag{2.2}$$

and the relation

$$t_n - \tau = t_{n-m} + \delta h \tag{2.3}$$

holds for the step points $t_n = t_0 + nh,\ n \in Z$.

By approximating the delayed argument and the integrand in (2.1) with linear interpolation, we can adapt a θ-method to (2.1) as follows:

$$u_{n+1} = u_n + h(1-\theta)f(t_n, u_n, v_n, G_n) + h\theta f(t_{n+1}, u_{n+1}, v_{n+1}, G_{n+1}), \tag{2.4}$$

where, $0 \le \theta \le 1$, u_n is an approximate value of $u(t_n)$, and

$$\begin{aligned} v_n &= (1-\delta)u_{n-m} + \delta u_{n-m+1}, & (2.5)\\ G_n &= \frac{h(1-\delta)^2}{2} g(t_n, t_{n-m}, u_{n-m}) + \frac{h(2-\delta^2)}{2} g(t_n, t_{n-m+1}, u_{n-m+1}) & \\ &\quad + h\sum_{k=2}^{m-1} g(t_n, t_{n-m+k}, u_{n-m+k}) + \frac{h}{2} g(t_n, t_n, u_n). & (2.6) \end{aligned}$$

As a result, the integral term of (2.1) is approximated with the trapezoidal rule. When $\theta = 1/2$ and $\delta = 0$, the formula (2.4)–(2.6) determines a method that belongs to a class of Runge-Kutta methods discussed in [7]. But, when $\theta \neq 1/2$, it gives another type of numerical method.

In the case of the test equation (1.1), the formula (2.4)–(2.6) is reduced to

$$\begin{aligned} u_{n+1} &= u_n + (1-\theta)\alpha u_n + \theta\alpha u_{n+1} \\ &\quad + \beta\Big[(1-\delta)(1-\theta)u_{n-m} + (\delta+\theta-2\delta\theta)u_{n-m+1} + \delta\theta u_{n-m+2}\Big] \\ &\quad + \gamma\Big[\frac{(1-\delta)^2(1-\theta)}{2} u_{n-m} + \frac{(2-\delta^2)(1-\theta) + (1-\delta)^2\theta}{2} u_{n-m+1} \\ &\quad + \frac{2-\delta^2\theta}{2} u_{n-m+2} + \sum_{k=3}^{m-1} u_{n-m+k} + \frac{1+\theta}{2} u_n + \frac{\theta}{2} u_{n+1}\Big], \end{aligned} \tag{2.7}$$

$$\alpha = h\lambda, \quad \beta = h\mu, \quad \gamma = h^2\kappa. \tag{2.8}$$

The characteristic equation of (2.7) is written as

$$\begin{aligned} z^{m+1} \; - \; & z^m - (1-\theta)\alpha z^m - \theta\alpha z^{m+1} \\ & -\beta\Big[(1-\delta)(1-\theta) + (\delta+\theta-2\delta\theta)z + \delta\theta z^2\Big] \\ & -\gamma\Big[\frac{(1-\delta)^2(1-\theta)}{2} + \frac{(2-\delta^2)(1-\theta)+(1-\delta)^2\theta}{2}z \\ & \quad + \frac{2-\delta^2\theta}{2}z^2 + \sum_{k=3}^{m-1} z^k + \frac{1+\theta}{2}z^m + \frac{\theta}{2}z^{m+1}\Big] = 0. \end{aligned} \tag{2.9}$$

Using (2.9) we define the sets $S_{\theta,m}^{(\delta)}$ and $S_\theta^{(\delta)}$ for $0 \le \delta < 1$ by

$$S_{\theta,m}^{(\delta)} = \{(\alpha,\beta,\gamma) \in \mathbb{C}^3 : \text{all the roots of (2.9) satisfy } |z| < 1\}, \tag{2.10}$$

$$S_\theta^{(\delta)} = \bigcap_{m\ge 1} S_{\theta,m}^{(\delta)}. \tag{2.11}$$

The set $S_\theta^{(\delta)}$ is an analogue of the δ-stability region of the θ-method [4].

When $z = 1$, the left hand side of (2.9) is equal to $-[\alpha+\beta+(m-\delta)\gamma]$. Hence, for any $m \ge 1$, $z = 1$ is not a root of (2.9) if and only if

(C_0) $\quad \alpha+\beta+(m-\delta)\gamma \ne 0$ for any $m \ge 1$.

Substituting $\sum_{k=3}^{m-1} z^k = (z^3 - z^m)/(1-z)$ into (2.9) and multiplying $(1-z)$, we get

$$z^m q(z) - p(z) = 0, \tag{2.12}$$

$$\begin{aligned} q(z) &= q_0 z^2 + q_1 z + q_2, & (2.13)\\ p(z) &= p_0 z^3 + p_1 z^2 + p_2 z + p_3, & (2.14) \end{aligned}$$

where

$$\begin{aligned} q_0 &= \theta\alpha + \frac{\theta}{2}\gamma - 1, \quad q_1 = (1-2\theta)\alpha + \frac{\gamma}{2} + 2, \\ q_2 &= -(1-\theta)\alpha + \frac{1-\theta}{2}\gamma - 1, \\ p_0 &= -\delta\theta\beta + \frac{\delta^2\theta}{2}\gamma, \quad p_1 = (3\delta\theta - \delta - \theta)\beta + \frac{-3\delta^2\theta + \delta^2 + 2\delta\theta + \theta}{2}\gamma, \\ p_2 &= (-3\delta\theta + 2\delta + 2\theta - 1)\beta + \frac{3\delta^2\theta - 2\delta^2 - 4\delta\theta - 2\delta^2 + 2\delta + 1}{2}\gamma, \\ p_3 &= (\delta\theta - \delta - \theta + 1)\beta + \frac{-\delta^2\theta + \delta^2 + 2\delta\theta - 2\delta - \theta + 1}{2}\gamma. \end{aligned}$$

Moreover, we set

$$r(z) = p(z)/q(z), \tag{2.15}$$

and consider the following conditions.

(a) $q(z) \neq 0$ for any $| z | \geq 1$.

($\hat{\mathbf{a}}$) $q(z) \neq 0$ for any $| z | > 1$.

(b) $| r(z) | < 1$ for any $| z | = 1$ with $z \neq 1$.

($\hat{\mathbf{b}}$) $| r(z) | \leq 1$ for any $| z | = 1$.

These are regarded as conditions for α, β, γ. We also write

(c) $(\alpha, \beta, \gamma) \in S_\theta^{(\delta)}$.

Under this notation, we can characterize $S_\theta^{(\delta)}$ as follows.

Theorem 2.1 *The following implications hold:*

$$(\mathrm{C}_0) \;\; \textit{and} \;\; \mathbf{(a)} \;\; \textit{and} \;\; \mathbf{(b)} \quad \Longrightarrow \quad \mathbf{(c)} \quad \Longrightarrow \quad (\hat{\mathbf{a}}) \;\; \textit{and} \;\; (\hat{\mathbf{b}}).$$

If, in addition,

(C_1) *$p(z)$, $q(z)$ have no common zero on $| z | = 1$,*

then **(c)** *implies* **(a)**.

Proof. Assume (C_0), **(a)** and **(b)**. We first show that $\hat{r}(z) = r(z)/z$ satisfies $| \hat{r}(z) | < 1$ for any $| z | \geq 1$ with $z \neq 1$.

The linear fractional transformation

$$z = \frac{w+1}{w-1} \tag{2.16}$$

maps $\mathrm{Re}\, w > 0$ conformally onto $| z | > 1$, with $w = \infty$ corresponding to $z = 1$. The function $\hat{R}(w) = \hat{r}[(w+1)/(w-1)]$ is represented in the form

$$\hat{R}(w) = \hat{P}(w)/\hat{Q}(w), \tag{2.17}$$

$$\hat{P}(w) = \Big[\gamma w^2 + (-2\beta + 2\delta\gamma - \gamma)w + 2(1-2\delta)\beta - 2\delta(1-\delta)\gamma\Big] \times \Big[w - (1-2\theta)\Big], \tag{2.18}$$

$$\hat{Q}(w) = (w+1)\Big\{\gamma w^2 + \Big[2\alpha - (1-2\theta)\gamma\Big]w - 2(1-2\theta)\alpha - 4\Big\}. \tag{2.19}$$

Then, it follows from **(a)** that $\hat{R}(w)$ is a bounded, holomorphic function in $\mathrm{Re}\, w > 0$. Hence, by the Phragmén-Lindelöf theorem (see, e.g., [8], p. 168), it follows from **(b)** that $| \hat{R}(w) | < 1$ for any $\mathrm{Re}\, w > 0$, which implies that $| \hat{r}(z) | < 1$ for any $| z | \geq 1$ with $z \neq 1$.

If $| z | \geq 1$ and $z \neq 1$, then

$$z^m q(z) - p(z) = q(z) z \Big[z^{m-1} - \hat{r}(z)\Big] \neq 0,$$

which, together with (C_0), implies **(c)**.

Assume **(c)**. If $q(z_0) = 0$ for some $| z_0 |> 1$, then there exists $\varepsilon > 0$ such that $C(z_0, \varepsilon) \subset \{| z |> 1\}$ and $q(z) \neq 0$ on $C(z_0, \varepsilon)$, where

$$C(z_0, \varepsilon) = \{z \in \mathbb{C} : | z - z_0 |= \varepsilon\}.$$

By Rouché's theorem, the polynomial $z^m q(z) - p(z)$ has a root in the interior of $C(z_0, \varepsilon)$ for sufficiently large m, which contradicts **(c)**. Therefore, **($\hat{\mathbf{a}}$)** holds.

Moreover, if $| r(z_0) |> 1$ for some $| z_0 |= 1$, then the equation $z^m = r(z)$ has a solution with $| z |> 1$ for sufficiently large m. This is verified by applying Proposition 7 of [11] to $\psi(z) = 1/r(z)$. In fact, there exists $\varepsilon > 0$ such that $| r(z) |> 1$ for any $z \in \overline{V_\varepsilon}$, where $V_\varepsilon = \{z \in \mathbb{C} : | z - (1+\varepsilon)z_0 |< \varepsilon\}$. Hence,

$$\rho = \max_{z \in \overline{V_\varepsilon}} | \psi(z) |< 1,$$

and $1 \in \mathbb{C} \backslash B(0, \rho)$, where $B(0, \rho) = \{z \in \mathbb{C} : | z |\leq \rho\}$. On the other hand, we have

$$\mathbb{C} \backslash B(0, \rho) \subset \bigcup_{m \geq 1} \{z^m \psi(z) \ : \ z \in V_\varepsilon\}, \tag{2.20}$$

by Proposition 7 of [11]. Since $| z |> 1$ for any $z \in V_\varepsilon$, it follows from (2.20) that $z^m = r(z)$ holds for some $m \geq 1$ and $| z |> 1$, which contradicts **(c)**. Therefore, **($\hat{\mathbf{b}}$)** holds.

It is easy to see that **($\hat{\mathbf{a}}$)** and **($\hat{\mathbf{b}}$)** imply **(a)** under the condition (C_1). □

3. Stability regions in the case $\delta = 0$

We consider the case $\delta = 0$. Since $q(1) = \gamma$, $z = 1$ satisfies $q(z) = 0$ if and only if $\gamma = 0$. We assume that $\gamma \neq 0$ for a while, and rewrite the conditions **(a)**, **($\hat{\mathbf{a}}$)**, **(b)**, **($\hat{\mathbf{b}}$)** by making use of the linear fractional transformation (2.16).

The function $R(w) = r[(w+1)/(w-1)]$ is represented in the form

$$\begin{aligned} R(w) &= P(w)/Q(w), & (3.1)\\ P(w) &= (\gamma w - 2\beta)\Big[w - (1 - 2\theta)\Big], & (3.2)\\ Q(w) &= \gamma w^2 + \Big[2\alpha - (1-2\theta)\gamma\Big]w - 2(1-2\theta)\alpha - 4. & (3.3) \end{aligned}$$

Hence, **(a)**, **($\hat{\mathbf{a}}$)**, **(b)**, **($\hat{\mathbf{b}}$)** are equivalent to

(A) $Q(w) \neq 0$ for any $\operatorname{Re} w \geq 0$,

($\hat{\mathbf{A}}$) $Q(w) \neq 0$ for any $\operatorname{Re} w > 0$,

(B) $| R(w) |< 1$ for any $\operatorname{Re} w = 0$,

($\hat{\mathbf{B}}$) $| R(w) |\leq 1$ for any $\operatorname{Re} w = 0$, respectively.

When $\alpha,\ \gamma$ are real, **(A)**, $(\widehat{\mathbf{A}})$ are reduced to

$$\gamma\Big[2\alpha-(1-2\theta)\gamma\Big]>0,\quad \gamma\Big[-4-2(1-2\theta)\alpha\Big]>0, \tag{3.4}$$

$$\gamma\Big[2\alpha-(1-2\theta)\gamma\Big]\ge 0,\quad \gamma\Big[-4-2(1-2\theta)\alpha\Big]\ge 0, \tag{3.5}$$

respectively. In addition, putting $w=\mathrm{i}y,\ y\in\mathbb{R}$, we have

$$\begin{aligned}
&|\,Q(w)\,|^2-|\,P(w)\,|^2\\
&\quad=4\,\mathrm{Im}\Big[(\alpha+\beta)\overline{\gamma}\Big]\,y^3+4\Big(|\,\alpha\,|^2-|\,\beta\,|^2+2\,\mathrm{Re}\,\gamma\Big)y^2\\
&\qquad+\Big\{16\,\mathrm{Im}\,\alpha+4(1-2\theta)^2\,\mathrm{Im}\Big[(\alpha+\beta)\overline{\gamma}\Big]\Big\}y\\
&\qquad\quad+|\,4+2(1-2\theta)\alpha\,|^2-|\,2(1-2\theta)\beta\,|^2\,.
\end{aligned} \tag{3.6}$$

When $\alpha,\ \beta,\ \gamma$ are real, it is reduced to

$$|\,Q(w)\,|^2-|\,P(w)\,|^2 = 4(\alpha^2-\beta^2+2\gamma)y^2+4\eta, \tag{3.7}$$

$$\eta = \Big[(1-2\theta)(\alpha+\beta)+2\Big]\Big[(1-2\theta)(\alpha-\beta)+2\Big]. \tag{3.8}$$

Hence, in this case, **(B)**, $(\widehat{\mathbf{B}})$ are equivalent to

$$\beta^2\le\alpha^2+2\gamma,\quad \eta>0, \tag{3.9}$$

$$\beta^2\le\alpha^2+2\gamma,\quad \eta\ge 0, \tag{3.10}$$

respectively.

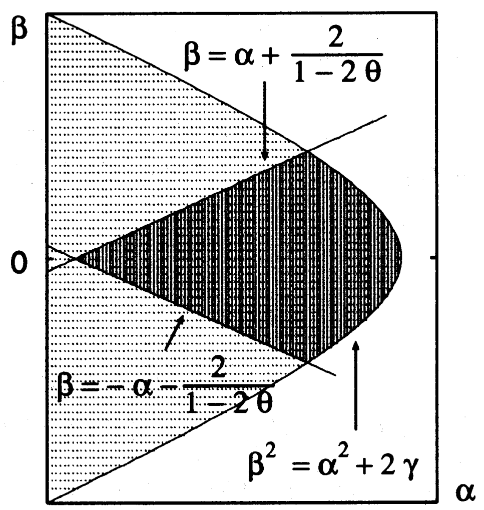

Fig. 2 γ-section of $S_\theta^{(0)}\cap\mathbb{R}^3$ $(0\le\theta<1/2)$

Let $\alpha < 0$ and $\gamma < 0$. The conditions (3.4), (3.9) are reduced to

$$\alpha > -\frac{2}{1-2\theta}, \quad \gamma > \frac{2\alpha}{1-2\theta}, \quad \beta^2 \leq \alpha^2 + 2\gamma, \quad |\beta| < \alpha + \frac{2}{1-2\theta}, \tag{3.11}$$

when $0 \leq \theta < 1/2$ (**Fig. 2**), and

$$\beta^2 \leq \alpha^2 + 2\gamma, \tag{3.12}$$

when $1/2 \leq \theta \leq 1$. If $\alpha(<0)$, β satisfy $\beta^2 \leq \alpha^2 + 2\gamma$ for $\gamma < 0$, then $\alpha + \beta < 0$, and (C_0) holds. Hence, by Theorem 2.1, these determine the region

$$S_\theta^{(0)} \cap \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 : \ \alpha < 0, \ \gamma < 0\}, \tag{3.13}$$

except for ambiguity of the boundary.

We now denote by Ω the set of all the triplicate (λ, μ, κ) for which the zero solution of (1.1) is asymptotically stable for any $\tau \geq 0$, i.e.,

$$\Omega = \{(\lambda, \mu, \kappa) \in \mathbb{C}^3 : (1.4), (1.5), (1.6) \text{ are satisfied}\}. \tag{3.14}$$

It is easy to see that

$$(\lambda, \mu, \kappa) \in \Omega \quad \Longrightarrow \quad (h\lambda, h\mu, h^2\kappa) \in \Omega \quad \text{for any } h > 0. \tag{3.15}$$

The following theorem shows that A-stable θ-methods possess a similar stability property to P-stability with respect to DIDEs.

Theorem 3.2 *If $1/2 \leq \theta \leq 1$, then $\Omega \subset S_\theta^{(0)}$.*

Proof. The inclusion $\Omega \cap \{\gamma = 0\} \subset S_\theta^{(0)}$ follows from the known result as in the case of DDEs (see, e.g., Theorem 2.6 in [6]). We consider the case $\gamma \neq 0$.

Let $(\alpha, \beta, \gamma) \in \Omega$. The condition (C_0) follows from (1.4). Moreover, it follows from (3.6) and $\mathrm{Im}[(\alpha+\beta)\overline{\gamma}] = 0$ that for $w = iy$, $y \in \mathbb{R}$,

$$\begin{aligned} |Q(w)|^2 - |P(w)|^2 &= \eta_0 y^2 + 2\eta_1 y + \eta_2, \\ \eta_0 = 4\big(|\alpha|^2 - |\beta|^2 &+ 2\,\mathrm{Re}\,\gamma\big), \quad \eta_1 = 8\,\mathrm{Im}\,\alpha, \\ \eta_2 = |2(1-2\theta)\alpha &+ 4|^2 - |2(1-2\theta)\beta|^2 . \end{aligned} \tag{3.16}$$

Since

$$\eta_2 = 16 + 16(1-2\theta)\,\mathrm{Re}\,\alpha + 4(1-2\theta)^2\big(|\alpha|^2 - |\beta|^2\big) \geq 16, \tag{3.17}$$

$$\begin{aligned} \eta_1^2 - \eta_0\eta_2 &\leq 64(\mathrm{Im}\,\alpha)^2 - 64\big(|\alpha|^2 - |\beta|^2 + 2\,\mathrm{Re}\,\gamma\big) \\ &= -64\big[(\mathrm{Re}\,\alpha)^2 + 2\,\mathrm{Re}\,\gamma - |\beta|^2\big], \end{aligned} \tag{3.18}$$

we have

$$|Q(w)| > |P(w)| \quad \text{for any } \mathrm{Re}\, w = 0, \tag{3.19}$$

which implies **(B)**.

When $\theta = 1/2$, it holds that

$$Q(w) = \gamma w^2 + 2\alpha w - 4 = -w^2 \Big[(2/w)^2 - \alpha(2/w) - \gamma \Big]. \tag{3.20}$$

Hence, **(A)** for $\theta = 1/2$ follows from (1.5).

The condition **(A)** for $\theta = 1/2$, together with (3.19), implies **(A)** for $1/2 < \theta \le 1$. In fact, if $Q(w) = 0$ has a solution with $\mathrm{Re}\, w \ge 0$ for some $1/2 < \theta \le 1$, then it follows from **(A)** for $\theta = 1/2$ that there exists $1/2 < \theta_0 \le \theta$ such that $Q(w) = 0$ for $\theta = \theta_0$ has a solution with $\mathrm{Re}\, w = 0$. But this is impossible by (3.19). □

4. Stability regions in the case $\delta \neq 0$

The same result as in Theorem 3.2 does not hold in the case $\delta \neq 0$. As a result, any θ-method cannot possess a similar stability property to GP-stability with respect to DIDEs.

Theorem 4.3 *If* $0 < \delta < 1$, *there exists* $(\alpha, \beta, \gamma) \in \Omega$ *which does not belong to* $S_\theta^{(\delta)}$.

Proof. The function $R(w) = r[(w+1)/(w-1)]$ can be written as

$$\begin{aligned} R(w) &= \tilde{P}(w)/\tilde{Q}(w), & (4.1)\\ \tilde{P}(w) &= \Big[\gamma w^2 + (-2\beta + 2\delta\gamma - \gamma)w + 2(1-2\delta)\beta - 2\delta(1-\delta)\gamma\Big] \\ &\quad \times \Big[w - (1-2\theta)\Big], & (4.2)\\ \tilde{Q}(w) &= (w-1)\Big\{\gamma w^2 + \Big[2\alpha - (1-2\theta)\gamma\Big]w - 2(1-2\theta)\alpha - 4\Big\}. & (4.3) \end{aligned}$$

When α, β, γ are real, we have for $w = iy$, $y \in \mathbb{R}$,

$$\begin{aligned} |\tilde{Q}(w)|^2 - |\tilde{P}(w)|^2 &= 4(y^2+1)\Big[(\alpha^2 - \beta^2 + 2\gamma)y^2 + \eta\Big] \\ &\quad + 4\delta(1-\delta)(2\beta - \delta\gamma)\Big[2\beta + (1-\delta)\gamma\Big]\Big[y^2 + (1-2\theta)^2\Big], & (4.4)\\ \eta &= \Big[(1-2\theta)(\alpha+\beta) + 2\Big]\Big[(1-2\theta)(\alpha-\beta) + 2\Big]. & (4.5) \end{aligned}$$

When $\alpha = -\sqrt{-2\gamma}$ and $\beta = 0$, (4.4) is a quadratic function of y and the coefficient of y^2 is given by

$$4\Big[-(1-2\theta)\sqrt{-2\gamma} + 2\Big]^2 - 4\delta^2(1-\delta)^2\gamma^2. \tag{4.6}$$

If $0 < \delta < 1$ and $-\gamma$ is sufficiently large, the value (4.6) is negative. This implies that **($\hat{\mathbf{b}}$)** does not hold near $(\alpha, \beta) = (-\sqrt{-2\gamma}, 0)$, a point on the hyperbola $\beta^2 = \alpha^2 + 2\gamma$, if $-\gamma$ is sufficiently large. Therefore, by Theorem 2.1, there are points in Ω which do not belong to $S_\theta^{(\delta)}$. □

In some cases, the region $S_\theta^{(\delta)} \cap I\!R^3$ is determined on the basis of Theorem 2.1. Let $1/2 \le \theta \le 1$, and assume that $\alpha < 0$ and $\gamma < 0$. Then, **(a)**, which does not depend on δ, is satisfied, and (C_0) holds if $\beta^2 \le \alpha^2 + 2\gamma$. Moreover, **(b)** is rewritten as

($\tilde{\mathbf{B}}$) $\quad |\, \tilde{Q}(w)\,| > |\, \tilde{P}(w)\,|$ for any $\mathrm{Re}\, w = 0$.

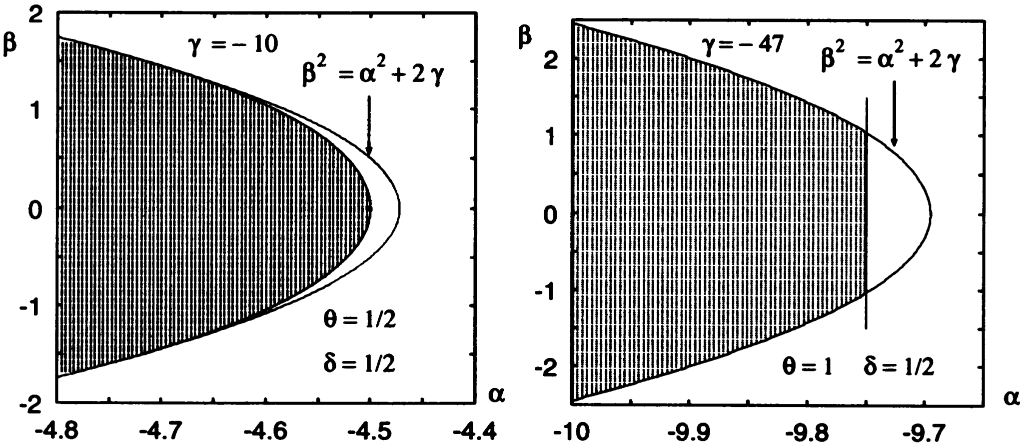

Fig. 3 Examples of γ-sections of $S_\theta^{(\delta)} \cap I\!R^3$ $(\delta = 1/2)$

In the case $\theta = 1/2$ (the trapezoidal rule), we have for $w = iy$, $y \in I\!R$,

$$|\, \tilde{Q}(w)\,|^2 - |\, \tilde{P}(w)\,|^2 = 4\Big[(y^2+1)(ay^2+4) + by^2\Big], \tag{4.7}$$
$$a = \alpha^2 - \beta^2 + 2\gamma, \tag{4.8}$$
$$b = \delta(1-\delta)(2\beta - \delta\gamma)\Big[2\beta + (1-\delta)\gamma\Big]. \tag{4.9}$$

From (4.7) it is easy to verify that **($\tilde{\mathbf{B}}$)** holds if and only if $a \ge 0$ and

$$a + b + 4 \ge 0, \quad \text{or} \quad \Big[a + b + 4 < 0 \text{ and } 16a > (a+b+4)^2\Big]. \tag{4.10}$$

When $\delta = 1/2$, this condition is represented as

$$\begin{cases} \beta^2 \le \alpha^2 + 2\gamma \quad \Big(\alpha^2 \ge \dfrac{\gamma^2}{16} - 2\gamma - 4\Big), \\ \beta^2 < \dfrac{1}{16}\left[15(\alpha^2+2\gamma) - \dfrac{\gamma^2}{16} - 4\right] \quad \Big(\alpha^2 < \dfrac{\gamma^2}{16} - 2\gamma - 4\Big). \end{cases} \tag{4.11}$$

In the case $\theta = 1$ (the backward Euler method), we have for $w = iy$, $y \in I\!R$,

$$|\, \tilde{Q}(w)\,|^2 - |\, \tilde{P}(w)\,|^2 = 4(y^2+1)(ay^2+c), \tag{4.12}$$
$$c = (2-\alpha)^2 - \beta^2 + \delta(1-\delta)(2\beta - \delta\gamma)\Big[2\beta + (1-\delta)\gamma\Big]. \tag{4.13}$$

The condition $(\tilde{\mathbf{B}})$ holds if and only if $a \geq 0,\ c > 0$, which is equivalent to

$$\beta^2 \leq \alpha^2 + 2\gamma, \quad \alpha < \frac{\gamma}{4} + 2, \tag{4.14}$$

when $\delta = 1/2$.

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